Uniform in Time (UiT) Convergence 相关结果

Zhizhou Liu (刘之洲)

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1 Results in [1]

由半群的 Strong Exponential Stability (SES) 以及 one-step weak error 的估计可以得到数值格式的 UiT weak convergence 及其收敛速率 [1]。这个条件看起来比较强，不过比较自然（不论是其证明还是理解上）。

Notation

- $t_n = n\delta$.
- X_t denotes the real solution and Y_{t_n} denotes the numerical solution of an SDE.
- $\mathscr{M}_n^\delta f(x) = \mathrm{E}^x[f(Y_{t_n}^\delta)]$ and $P_t f(x) = \mathrm{E}^x[f(X_t)]$.

1.1 A General UiT Weak Convergence Theorem

Assumption 1.1. (i) SES. There exists $K_0, \lambda > 0$ such that

$$\|P_t f\|_{C_b^2} \leq K_0 \|f\|_{C_b^2} \cdot e^{-\lambda t} \tag{1}$$

(Unsolved) Question 1. 为什么 $P_t f(x)$ 梯度、二阶梯度存在？如果存在是否有界？即 $P_t : C_b^2(\mathbb{R}^n) \to C_b^2(\mathbb{R}^n)$?

(ii) Local consistency & a-priori (uniform) control. There exists $\phi, \Phi : \mathbb{R}^N \times \mathbb{R}^+ \to [0, \infty)$ and $K_1 > 0$ such that

$$|\mathrm{E}^x[f(Y_\delta) - f(X_\delta)]| \leq K_1 \|f\|_{C_b^2} \cdot \phi(x, \delta) \tag{2}$$

for all $x \in \mathbb{R}^N$, $f \in C_b^2(\mathbb{R}^N)$ and $\delta > 0$. Also

$$\sup_{n\in\mathbb{N}} \mathrm{E}^x[\phi(Y_{t_n}^\delta, \delta)] = \sup_{n\in\mathbb{N}} (\mathscr{M}_n^\delta \phi)(x, \delta) \le \Phi(x, \delta) \tag{3}$$

for all $x \in \mathbb{R}^N$ and $\delta > 0$.

Remark 1.2. 这里需要注意的是，$\|\cdot\|_{C_b^2}$ 并非一个真正的 norm, 而是一个所谓的 seminorm. 它的具体定义为 for $f \in C^2(\mathbb{R}^N)$,

$$\|f\|_{C_b^2} \overset{\text{def}}{=} \sup_x \left(|\nabla f(x)| + \|\nabla^2 f(x)\|\right),$$

where $\|A\| \overset{\text{def}}{=} \sqrt{\sum_{i,j} |a_{ij}|^2}$ denotes the Frobenius norm of a matrix. 因此，SES 条件的意思并非 P_t 按照指数速率收敛到 0，而是以指数速率收敛到一个常数：

$\|f\|_{C_b^2} = 0$ iff $\nabla f(x) = 0$ (zero vector) and $\nabla^2 f(x) = 0$ for all $x \in \mathbb{R}^N$. 由中值定理可知，f 必为一个常值映射。

我们这里的 SES 条件并没有指明 P_t 将会收敛到的半群映射。与 SES 条件表达类似含义的是 [5, Theorem 1.1(ii)] 的结果 [1] ：

$$\|P_t f - \mu(f)\|_{L^q} \le C_q \cdot e^{-\alpha t/q} \|f\|_{L^q}, \tag{4}$$

where $q > 1$ and $\alpha, C > 0$ independent of x_0 and t. 这里 $\mu(f)$ 正是一个常值半群映射，且收敛速度也是指数阶。

(Unsolved) Question 2. Norm $\|\cdot\|_{L^q}$ 与 seminorm $\|\cdot\|_{C_b^2}$ 之间是否存在不等式关系?

更多关于 SES 条件的分析可以参看 [2, Section 4].

（所有框内文字均为补充注解，不重要，可跳过。）以下是与猜测相关的结果（并非对于其的证明）。利用中值定理，

$$|P_t f(x) - P_t f(y)| \le \|\nabla P_t f\| \cdot |x - y|.$$

由 SES 条件(1), 就有

$$|P_t f(x) - P_t f(y)| \le K_0 \|f\|_{C_b^2} \cdot e^{-\lambda t} |x - y|. \tag{5}$$

如果此时 x, y 是随机变量 X, Y, 那么有

$$\begin{aligned} |\mathrm{E}[P_t f(X) - P_t f(Y)]| &\le \mathrm{E}\,|P_t f(X) - P_t f(Y)| \\ &\le K_0 \|f\|_{C_b^2} \cdot e^{-\lambda t}\, \mathrm{E}\,|X - Y|, \end{aligned}$$

这里的 E 是对 X, Y 的同时的（不妨假设他们在相同的样本空间中）。如果 $X = x$ a.s., Y 服从不变测度 μ（假设存在不变测度）对应的分布，那么就有

$$|P_t f(x) - \mu(f)| \le K_0 \|f\|_{C_b^2} \cdot e^{-\lambda t} \int |x - y|\, \mu(\mathrm{d}y).$$

只需 $\int |y| \mu(\mathrm{d}y) < \infty$, 右侧就有界，但右侧无法得到 $\|f\|_{L^q}$ 项。

言归正传，under Assumption 1.1，我们可以得到 UiT weak convergence 及其收敛速率。

[1] 注意，得到这个结果需要假设 [5, (H4)] 中的 p 严格大于 2，这和我们一直使用的假设 $p = 2$ 不符合，且没有强弱关系。

Theorem 1.3. *With the notation we introduced so far, under Assumption 1.1, the following bound bolds for any $f \in C_b^2(\mathbb{R}^N)$ and $\delta > 0$ small enough:*

$$\sup_{n\in\mathbb{N}} \left| \mathrm{E}^x f(Y_{t_n}^\delta) - \mathrm{E}^x f(X_{t_n}) \right| \le \frac{K \|f\|_{C_b^2} \cdot \Phi(x,\delta)}{1 - e^{-\lambda\delta}}, \tag{6}$$

with $K \stackrel{def}{=} K_1(K_0 \vee 1)$.

以下是 observations，用以说明 Assumption 1.1 中条件的作用。定义 $Y_{t_n}^{\delta,k}$ denotes the Markov chain (indexed by n) that evolves according to the time discretisation until time t_k and then evolves according to SDE. 那么 $X_{t_n} = Y_{t_n}^{\delta,0}, Y_{t_n}^\delta == Y_{t_n}^{\delta,n}$. 有了这样的一头一尾，我们就将欲估计的式子改写为如下 (telescopic) sum 的形式

$$\mathrm{E}^x f(Y_{t_n}^\delta) - \mathrm{E}^x f(X_{t_n}) = \sum_{k=1}^n \left[\mathrm{E}^x f(Y_{t_n}^{\delta,k}) - \mathrm{E}^x f(Y_{t_n}^{\delta,k-1}) \right].$$

事实上，该 observation 可以直接被用以证明有终止时刻（not UiT）的 general weak convergence theorem，参见 [4, Theorem 2.6]. 彼处通过直接的假设使得该 observation 能直接用到证明中；但在我们的假设条件下，该观察无法直接用于证明。观察 $Y_{t_n}^{\delta,k}$ 与 $Y_{t_n}^{\delta,k-1}$ 间的差距，(i) 发现它们在 $k-1$ 次前都是一样的，(ii) 在 t_k 时刻，$Y_{t_n}^{\delta,k}$ 遵照数值格式演化；而 $Y_{t_n}^{\delta,k-1}$ 按 SDE 演化；它们的演化初值都是 $Y_{t_{k-1}}^\delta$。(iii) 在 t_k 时刻之后，它们都按照 SDE 演化，只是初始位置不同。对于 (ii)，我们需要 one-step weak error 的估计(2)，并要求一个一直上界(3). 对于 (iii)，由(5) 可知其应该是可和的（注意(5)是 SES 的推论）。
这样，从 intuition 的角度可以理解我们需要 Assumption 1.1 的原因。在 [4, Theorem 2.6] 中，作者直接假设了

$$\mathrm{E}^x f(Y_{t_n}^{\delta,k}) - \mathrm{E}^x f(Y_{t_n}^{\delta,k-1})$$

的有界性，从而得到可和性；但在我们的假设下，$Y_{t_n}^{\delta,k}$ 这样的过程是无从定量分析的，因此需要另辟蹊径。

Remark 1.4. Assumption 1.1 的每一条对于 UiT weak convergence 而言都是定性上必要的，[3] 给出了它们不能互相推出，以及缺少其中某个条件后，结论不成立的例子。

Proof of Theorem 1.3. 将

$$\mathrm{E}^x f(Y_{t_n}^\delta) - \mathrm{E}^x f(X_{t_n}) = \mathscr{M}_n^\delta f(x) - P_{t_n} f(x)$$

添补项为

$$G_n(x) \stackrel{\text{def}}{=} \mathscr{M}_n^\delta f(x) - P_{t_n} f(x) = \underbrace{\mathscr{M}_n^\delta f(x) - \mathscr{M}_1^\delta (P_{t_{n-1}} f)(x)}_{A_n(x)} + \underbrace{\mathscr{M}_1^\delta (P_{t_{n-1}} f)(x) - P_{t_n} f(x)}_{B_n(x)}. \tag{7}$$

[2] $A_n(x)$ 表示从 x 开始由数值格式演化与第 1 步数值格式后 SDE 演化的差，相当于 $n-1$ 步的 weak error; $B_n(x)$ 表示第 1 步数值格式后 SDE 演化与由 SDE 演化的差，可由一步 weak error 配合 SES 条件进行估计。具体而言：

$$\begin{aligned} |B_n(x)| &= \left| \mathrm{E}^x \left[P_{t_{n-1}} f(Y_{t_1}^\delta) - P_{t_{n-1}} f(X_\delta) \right] \right| \\ &\overset{(2)}{\le} K_1 \left\| P_{t_{n-1}} f \right\|_{C_b^2} \cdot \phi(x,\delta) \\ &\overset{(1)}{\le} K_1 K_0 \|f\|_{C_b^2} \cdot \phi(x,\delta) e^{-\lambda(n-1)\delta}. \end{aligned}$$

[2]此式原论文 [1] 有笔误，先作用的半群应写在前面。

而

$$
\begin{aligned}
|A_n(x)| &= \left|\mathrm{E}\left[\mathscr{M}_{n-1}^{\delta} f(Y_{t_1}^{\delta}) - P_{t_{n-1}} f(Y_{t_1}^{\delta}) \mid Y_0^{\delta} = x\right]\right| \\
&= \left|\mathrm{E}[G_{n-1}(Y_{t_1}^{\delta}) \mid Y_0^{\delta} = x]\right|
\end{aligned}
$$

故尝试数学归纳法。通过计算 $n = 2$ 的情况（略），猜测得到

$$
\left|\mathscr{M}_n^{\delta} f(x) - P_{t_n} f(x)\right| \leq K \|f\|_{C_b^2} \sum_{k=0}^{n-1} \mathscr{M}_k^{\delta} \phi(x, \delta) e^{-\lambda(n-1-k)\delta},
$$

where $K = K_1(K_0 \vee 1)$ for all $x \in \mathbb{R}^N$, 再使用归纳法证明这个结论。最后另 $n \to \infty$, 使用(3)与等比数列和极限公式，就得到了最终结果。 □

Remark 1.5. 在 Theorem 1.3 的证明中，我们发现这里并没有用到 $\|\cdot\|_{C_b^2}$ 的定义，也就是说，这个定理对于任意的 seminorm 均成立，只要结论中的 seminorm 与 Assumption 1.1 中的对应即可。所选择的 seminorm 需要使得原半群满足条件 (1); 数值格式半群满足 (2) 和 (3). 在验证 global Lipschitz 条件的 Euler 格式时，条件(2) 似乎只对 $\|\cdot\|_{C_b^2}$ 才容易验证。如果如此，那么我们在对应的 SES 条件 (1) 中也只能使用 $\|\cdot\|_{C_b^2}$, 但这个条件是很难验证的。[1] 中给出的系数充分条件也仅仅只是对 $b(x) = -x - x^3$ 这样的例子正确，对 $b(x) = x - x^3$ 就不正确了。

1.2 UiT Weak Convergence for Explicit Euler Scheme with Global Lipschitz Coefficient: Application of Theorem 1.3

Lemma 1.6. *Assume that condition (2) is satisfied, and that the function $\phi(x, \delta)$ defined can be written in the form $\phi(x, \delta) = \delta^{\alpha} g(x) + \delta^{\beta}$ for some $\alpha, \beta > 0$ and $g : \mathbb{R}^n \to \mathbb{R}$ such that*

$$
\mathscr{M}_1^{\delta} g(x) \leq \epsilon g(x) + c,
$$

for some $\epsilon \in (0, 1)$ and $c > 0$ (both may depend on δ). Then condition (3) is satisfied with

$$
\Phi(x, \delta) = \delta^{\alpha} g(x) + \frac{c\delta^{\alpha}}{1 - \epsilon} + \delta^{\beta}.
$$

以下 Lemma 是我们得以进行 weak error analysis: condition (2) 的关键。

Lemma 1.7. *Let x_t, y_t be two Feller process with Markov semigroup P_t and Q_t, respectively. Denoting by $\mathscr{L}^P$ and $\mathscr{L}^Q$ the corresponding infinitesimal generator, the following identity holds,*

$$
P_t f(x) - Q_t f(x) = \int_0^t \mathrm{E}^x[(\mathscr{L}^P - \mathscr{L}^Q) P_{t-s} f(y_s)] \mathrm{d}s,
$$

for every $f \in C_b^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$ and $t \geq 0$.

Proof. Followed by Itô formula, we can write

$$
\mathrm{E}^x[\phi(t, y_t)] = \phi(0, y_0) + \int_0^t \mathrm{E}^x\left[\partial_s \phi_s(y_s) + \mathscr{L}^Q \phi_s(y_s)\right] \mathrm{d}s,
$$

for every $\phi(t,\cdot)\in C_b^2(\mathbb{R}^n)$. Fix $t\geq 0$ and choose $\phi_s=P_{t-s}f$ with $f\in C_b^2(\mathbb{R}^n)$ (需要 Question 1 的结论), for every $s\in[0,t]$, we obtain

$$\begin{aligned}Q_tf(x)=\mathrm{E}^x[f(y_t)]&=P_tf(x)+\int_0^t \mathrm{E}^x\left[\partial_s P_{t-s}f(y_s)+\mathscr{L}^Q P_{t-s}f(y_s)\right]\mathrm{d}s\\&=P_tf(x)+\int_0^t \mathrm{E}^x\left[-\mathscr{L}^P P_{t-s}f(y_s)+\mathscr{L}^Q P_{t-s}f(y_s)\right]\mathrm{d}s,\end{aligned}$$

where the equality follows by the definition of $\mathscr{L}^P$,

$$\begin{aligned}\partial_s P_{t-s}f(x)&=\lim_{h\to 0}\frac{P_{t-(s-h)}f(x)-P_{t-s}f(x)}{-h}\\&=-\lim_{h\to 0}\frac{P_h(P_{t-s}f)(x)-P_{t-s}f(x)}{h}\\&=-\mathscr{L}^P P_{t-s}f(x).\end{aligned}$$

This gives the statement. □

To be clear, let $\{Y_{t_n}^\delta\}$ be the explicit Euler scheme of the SDE

$$\mathrm{d}X_t=b(X_t)\mathrm{d}t+\sigma(X_t)\mathrm{d}W_t,\quad X_0=x, \tag{8}$$

i.e.,

$$Y_{t_{n+1}}^\delta=Y_{t_n}^\delta+b(Y_{t_n}^\delta)\delta+\sigma(Y_{t_n}^\delta)\Delta W_{t_n},\quad Y_0^\delta=x, \tag{9}$$

where $t_n=n\delta$ and $\Delta W_{t_n}=W_{t_{n+1}}-W_{t_n}$.

The idea of analyzing the weak error between X_t and Y_t comes from Lemma 1.7. It is well-known that the generator of X_t can be written down explicitly via Itô formula. As for the numerical process Y_{t_n}, we apply the following interpolation technique.

Let Y_t^δ defined by

$$\mathrm{d}Y_t^\delta=b(Y_{t_{n(t)}}^\delta)\mathrm{d}t+\sigma(Y_{t_{n(t)}})\mathrm{d}W_t \tag{10}$$

with $t_{n(t)}=t_i$ for $t\in[t_i,t_{i+1})$. Apply Itô formula on each interval and then summing up, we shall obtain the following Itô-formed formula, see [3, Lemma 3.5]

$$\begin{aligned}\phi(t,Y_t^\delta)=\phi(0,Y_0^\delta)+&\int_0^t\left(\partial_s\phi(s,Y_s^\delta)+\mathscr{L}_{\left(Y_{t_{n(s)}}^\delta\right)}\phi(s,Y_s^\delta)\right)\mathrm{d}s\\&+\int_0^t\nabla\phi(s,Y_s^\delta)^\top\sigma(Y_{t_{n(s)}}^\delta)\mathrm{d}W_s.\end{aligned}$$

Here ∇ is gradient w.r.t. the spatial variables and

$$(\mathscr{L}_{(v)}f)(x)\stackrel{\text{def}}{=}\sum_{i=1}^n b^i(v)\partial_i f(x)+\frac{1}{2}\sum_{i,j=1}^n(\sigma(v)\sigma(v)^\top)_{ij}\partial_{ij}^2 f(x).$$

Note that $\mathscr{L}f(x)=\mathscr{L}_{(x)}f(x)$. Therefore by Lemma 1.7, we can write

$$\mathrm{E}^x[f(X_t)]-\mathrm{E}_x[f(Y_t^\delta)]=\mathrm{E}^x\left[\int_0^t\left(\mathscr{L}_{(Y_s^\delta)}-\mathscr{L}_{\left(Y_{t_{n(s)}}^\delta\right)}\right)(P_{t-s}f)(Y_s^\delta)\mathrm{d}s\right] \tag{11}$$

for $f\in C_b^2(\mathbb{R}^n), t\geq 0$. We shall choose $t=\delta$ in the proof of the below theorem to obtain the local consistency (2).

Theorem 1.8. *Under global Lipschitz assumption for b, σ with Lipschitz constant c_1, c_2 respectively together with the boundedness of σ by M, conditions (2) is satisfied by the explicit Euler scheme Y_t^δ with $\phi(x, \delta) = |x|\,\delta^2 + \delta^{3/2}$.*

If in addition, there exists constants $b_0, b_1 \geq 0$ such that

$$\langle b(x), x\rangle \leq -b_0\,|x|^2 + b_1,$$

then by Lemma 1.6, (3) is satisfied.

If in addition, the SES condition (1) is satisfied as well, then by Theorem 1.3, there exists $\tilde{K} > 0$ such that

$$\sup_{t\geq 0}\left|\mathrm{E}^x[f(X_t)] - \mathrm{E}^x[f(Y_t^\delta)]\right| \leq \tilde{K}\,\|f\|_{C_b^2}\cdot\left(\delta\,|x| + \delta^{1/2}\right)$$

for any $f \in C_b^2(\mathbb{R}^n)$ and $\delta > 0$ small enough. That is, the explicit Euler scheme (9) is a UiT approximation of the corresponding SDE (8).

Proof. Choose $t = \delta$ in (11), then

$$\mathrm{E}^x[f(X_\delta)] - \mathrm{E}_x[f(Y_t^\delta)] = \mathrm{E}^x\left[\int_0^\delta \left(\mathscr{L}_{(Y_s^\delta)} - \mathscr{L}_{(x)}\right)(P_{\delta-s}f)(Y_s^\delta)\mathrm{d}s\right].$$

By the definition of the operator $\mathscr{L}_{(v)}$, we have

$$\begin{aligned}&\mathrm{E}^x[f(X_\delta)] - \mathrm{E}_x[f(Y_t^\delta)]\\ =&\,\mathrm{E}^x\Bigg[\int_0^\delta \left\langle b(Y_s^\delta) - b(x), \nabla P_{\delta-s}f(Y_s^\delta)\right\rangle\\ &\quad + \frac{1}{2}\sum_{i,j=1}^n(\sigma(Y_s^\delta)\sigma(Y_s^\delta)^\top - \sigma(x)\sigma(x)^\top)\partial_{ij}^2 P_{\delta-s}f(Y_s^\delta)\mathrm{d}s\Bigg].\end{aligned}$$

Due to the global Lipschitz assumptions of b, σ and the boundedness of σ, we have

$$\begin{aligned}&\mathrm{E}^x[f(X_\delta)] - \mathrm{E}_x[f(Y_t^\delta)]\\ \leq&\, C\,\mathrm{E}^x\left[\int_0^\delta\left(\left|\nabla P_{\delta-s}f(X_s^\delta)\right| + \left\|\nabla^2 P_{\delta-s}f(X_s^\delta)\right\|\right)\left|X_s^\delta - x\right|\mathrm{d}s\right].\end{aligned}$$

If the answer to Question 1 is positive, then $\|P_t f\|_{C_b^2} \leq K\,\|f\|_{C_b^2}$ for some constant K. Moreover, by construction of the scheme, the local strong error is bounded by

$$\mathrm{E}^x[\left|X_s^\delta - x\right|] \leq |b(x)|\,s + M\,\mathrm{E}\,|W_s| \leq C\left(|x|\,s + s^{1/2}\right),$$

where we used $|b(x)| \leq C(|x| + 1)$ deduced by the global Lipschitz of b and Hölder's inequality. After completing the integration condition, (2) follows.

The remaining part of the proof is straightforward. Term $1 - e^{-\lambda\delta}$ disappear here because we do not care about the coefficient $\tilde{K}$ in this theorem so that we can replace it with its equivalence small value(等价无穷小), which is δ, as δ is small enough. ☐

Remark 1.9. [1, Section 2.1] also gives sufficient coefficient-criteria for the SES condition. However, the criteria only works for $b(x) = -x - x^3$ ([1, Example 2.5]) but not for $b(x) = x - x^3$ (easily checked).

1.3 UiT Weak Convergence for Implicit Euler Scheme with One-sided Lipschitz Coefficient: Modification of SDE

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